An improved model for bubble formation using the boundary-integral method

Zongyuan Xiao, Reginald B.H. Tan*

Department of Chemical and Biomolecular Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore

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Abstract

A model for bubble formation from a single submerged orifice is developed using the boundary-integral method. Since the flow field is assumed to be irrotational, potential-flow theory is used to predict the growth of the bubble. The effects of the surface tension and the liquid circulation around the bubble are included in the calculation. Predictions of bubble shape, chamber pressure and the effect of surface tension are presented to compare with reported experimental data.

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1. Introduction

Over the past decades, numerous theoretical and experimental studies have been reported in the field of bubble formation. In addition, many models have been developed to predict bubble formation, such as those by Marmur and Rubin (1976), Pinczewski (1981), Tsuge and Hibino (1983), Tan and Harris (1986) and Hooper (1986).

In the model proposed by Hooper (1986), the boundary-integral method, a popular and successful numerical technique, was used to predict bubble formation. The boundary-integral method is based on Green’s formula that enables us to reformulate the potential problem as the solution of a Fredholm integral equation. This formulation has the effect of reducing the dimension of the problem by one, which has the advantage of rendering the technique computationally efficient, yet rigorous. Some applications of this method can be found in Blake et al. (1986, 1987), Bonnet (1995) and Power (1995).

2. Model development and numerical method

The system under consideration is shown in Fig. 1. Gas is pumped into a chamber at a constant flow rate, $Q$, and bubbles out through a small orifice into a bath of liquid. The viscosity of the liquid is assumed negligible and the flow is assumed to be irrotational. Therefore, a velocity potential exists, $\phi = \nabla^2 \phi$, with Laplace’s equation

$$\nabla^2 \phi = 0.$$  \hfill (1)

Then Bernoulli integral is applied between the liquid side of the bubble surface and a point in the liquid chosen at a large distance from the orifice on the plane $z = 0$,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|u|^2 + \frac{P_l}{\rho_l} + gz = \frac{P_o}{\rho_l}$$ \hfill (2)

where $P_o = P_\infty + \rho_l g H$ is the hydrostatic pressure at the orifice, $P_\infty$ is the system pressure and $H$ is the total height of liquid above the orifice.

It is convenient to express the dynamic boundary condition in terms of a material derivative by employing the
Fig. 1. Schematic diagram of the bubble system.

identity

\[
\frac{D \phi}{Dr} = \frac{\partial \phi}{\partial t} + |u|^2
\]

Thus

\[
\frac{D \phi}{Dr} = \frac{1}{2} |u|^2 - g z + \frac{P_o - P_l}{\rho l}.
\] (3)

The liquid phase pressure at any point on the bubble surface, \(P_l\), is related to pressure within the bubble, \(P_b\), by

\[P_l = P_b - \sigma \kappa\] (4)

where \(\sigma\) is the surface tension coefficient and \(\kappa\) the local curvature of the bubble surface.

2.1. Thermodynamic equations for the gas flow

Conservation of mass on the chamber yields

\[V_c \frac{dp_c}{dr} = \rho_a Q - \rho_c q\] (5)

where \(\rho_a\) and \(\rho_c\) are the gas densities at supply and chamber conditions, respectively. From the first law of thermodynamics applied to the bubble, it follows that

\[
\frac{d}{dt} (E_b) = -P_b \frac{dV_b}{dr} + \Delta Q + \Delta W
\] (6)

where \(E_b\) is the internal energy of the gas in the bubble.

Assuming the gas satisfies the ideal gas law,

\[E_b = \frac{P_b V_b}{\gamma - 1}\]

where \(\gamma\) is the adiabatic exponent. \(\Delta Q\) is the heat added, and assuming adiabatic behavior within the bubble and chamber, it follows that:

\[\Delta Q = \frac{P_c q}{\gamma - 1}\]

\(\Delta W\) is the work done externally and is given by

\[\Delta W = P_c q\]

Substituting these equations into Eq. (6) it follows that:

\[V_b \frac{dP_b}{dr} + \gamma P_b \frac{dV_b}{dr} = \gamma P_c q\]

(7)

Similarly for the chamber alone, we derive

\[V_c \frac{dP_c}{dr} = \gamma P_a Q - \gamma P_c q\]

(8)

The volumetric flow through the orifice, \(q\), as follows:

\[q = \frac{\rho_b}{\rho_c} \frac{dV_b}{dr}\] (9)

Substituting the above equation into Eq. (8) and integrating equation, it follows:

\[P_c(t) - P_c(0) = \frac{\gamma P_a}{V_c} Qt - \frac{\gamma P_a P_b}{\rho_c V_c} (V_b(t) - V_b(0))\] (10)

Assuming the density is constant \(\rho_b = \rho_c\) and \(P_a = P_c\), the above equation becomes

\[P_c(t) = P_c(0) + \frac{c_0^2 \rho_c}{V_c} [Qt - V_b(t) + V_b(0)]\] (11)

where \(c_0^2 = \gamma RT_c\), the sound speed assumed to be constant.

The orifice equation that relates the volumetric flow through the orifice, \(q\), to the pressure difference across the orifice as follows:

\[q = k (\pi R_o^2) \sqrt{\frac{P_c - P_b}{\rho_c}}\] (12)

where \(k\) is the orifice coefficient.

Initially, if we assume \(q = Q\), it follows that:

\[P_b(0) = P_c(0) - \frac{Q^2 \rho_c}{k^2 (\pi R_o^2)^2}\] (13)

From Eqs. (11) (12) and (13), we obtain

\[P_b(t) = P_b(0) + \frac{c_0^2 \rho_c}{V_c} [Qt - V_b(t) + V_b(0)]\]

\[+ \frac{\rho_c}{k^2 (\pi R_o^2)^2} \left[ Q^2 - \left( \frac{dV_b}{dr} \right)^2 \right]\] (14)

2.2. Curvature of bubble surface

In the absence of liquid cross-flow across the orifice, the bubble can be assumed to be axisymmetric, and we use a number of two-dimensional \((r, z)\) elements to represent it.
According to the Young–Laplace equation, the local curvature of certain point on the surface is defined as

\[
\kappa = \frac{1}{R_1} + \frac{1}{R_2}
\]

where \( R_1 \) and \( R_2 \) are principal radii of curvature, on vertical and horizontal planes, respectively. From analytical geometry, the principal radii of curvature may be represented by the following equations (Gray, 1998):

\[
\frac{1}{R_1} = \frac{z''}{(1 + z'^2)^{3/2}} \quad \text{and} \quad \frac{1}{R_2} = \frac{z'}{r(1 + z'^2)^{1/2}}
\]

where, \( r \) and \( z \) represent horizontal and vertical positions on the profile, and \( z' \) and \( z'' \) denote the first and second derivatives with respect to \( r \) which can be obtained through curve fitting written in the form \( z = z(r) \) along the bubble surface.

2.3. Volumetric growth rate of bubble

With the values of the normal velocity \( \frac{\partial \phi}{\partial n} \) at a number of points on bubble surface \( S_b \), the bubble volumetric growth rate can be obtained by the integral

\[
\frac{dV_b}{dr} = 2\pi \int_{S_b} \frac{\partial \phi}{\partial n} r \, ds.
\]

2.4. Normal velocity

For any sufficiently smooth function \( \phi \), which satisfies Laplace’s equation in a domain \( \Omega \) having a piecewise smooth surface \( S \), Green’s integral formula can be written as (Brebbia, 1978; Jaswon and Symm, 1977):

\[
e(p)\phi(p) + \int_S \phi(q) \frac{\partial}{\partial n} G(p, q) \, dS
\]

\[
= \int_S \frac{\partial}{\partial n} (\phi(q)) G(p, q) \, dS
\]

where \( p \in \Omega + S, q \in S, \frac{\partial}{\partial n} \) is the normal derivative outward from \( S \), \( G(p, q) \) is Green’s function

\[
G(p, q) = \frac{1}{|p - q|}
\]

and

\[
e(p) = \begin{cases} 4\pi, & \text{if } p \in \Omega \\ 2\pi, & \text{if } p \in S \end{cases}
\]

2.5. Axisymmetric bubbles

As the bubble is assumed to be axisymmetric, a cylindrical coordinate system is used. Two-dimensional integrals are reduced further to one dimension by integrating through the polar angle. In this case the Green’s functions involve complete elliptic integrals.

In order to solve the problem of the two-dimensional bubble computationally, firstly the surface is represented by \( n \) points which divide the surface into \((n-1)\) elements. Then an isoparametric linear approximation is used for the surface and the functions. Thus on each segment, \( S_j (j = 1, L, n - 1) \), we have

\[
r(\xi) = r_j (1 - \xi) + r_{j+1} \xi
\]

\[
z(\xi) = z_j (1 - \xi) + z_{j+1} \xi
\]

\[
\phi(\xi) = \phi_j (1 - \xi) + \phi_{j+1} \xi
\]

\[
\frac{\partial \phi}{\partial n}(\xi) = \frac{\partial \phi_j}{\partial n} (1 - \xi) + \frac{\partial \phi_{j+1}}{\partial n} \xi
\]

where the parameter \( \xi \) is in the range \((0, 1)\).

Collocation points are chosen to be these \( n \) points on the surface, yielding \( n \) equations with the following form:

\[
2\pi \phi_i + \sum_{j=1}^{n-1} \int_{S_j} \left[ (1 - \xi) \frac{\partial \phi_j}{\partial n} + \xi \frac{\partial \phi_{j+1}}{\partial n} \right] \left( \frac{1}{|p_i - q_j|} \right) \, ds
\]

\[
= \sum_{j=1}^{n-1} \int_{S_j} \left[ (1 - \xi) \frac{\partial \phi_j}{\partial n} + \xi \frac{\partial \phi_{j+1}}{\partial n} \right] \left( \frac{1}{|p_i - q_j|} \right) \, ds
\]

\((i = 1, L, n)\) (21)

where \( p_i \) is the \( i \)th point chosen as collocation point with cylindrical coordinates \( p_i = (r_i, z_i, 0) \) and \( q_j \) is any point on the segment \( S_j \) with cylindrical coordinates \( q_j = (r(\xi), z(\xi), \alpha) \).

After assembly, the set of equations has the following matrix structure:

\[
A \phi = B \frac{\partial \phi}{\partial n}
\]

(22)

where \( A \) and \( B \) are \( n \times n \) matrices with the following values of items \( a_{ij} \) and \( b_{ij} \)

\[
a_{ij} = 2\pi \delta_{ij} + \int_{S_j} (1 - \xi) \frac{\partial}{\partial n} \left( \frac{1}{|p_i - q_j|} \right) \, ds + \int_{S_{j-1}} \xi \frac{\partial}{\partial n} \left( \frac{1}{|p_i - q_{j-1}|} \right) \, ds
\]

\[
b_{ij} = \int_{S_j} (1 - \xi) \frac{1}{|p_i - q_j|} \, ds + \int_{S_{j-1}} \xi \frac{1}{|p_i - q_{j-1}|} \, ds
\]

\((i = 1, L, n)\) (23)

\((j = 1, L, n)\) (24)

\( \phi \) and \( \frac{\partial \phi}{\partial n} \) are vectors as follows:

\[
\phi = [\phi_1, \phi_2, LL, \phi_n]^T
\]

\[
\frac{\partial \phi}{\partial n} = \left[ \frac{\partial \phi_1}{\partial n}, \frac{\partial \phi_2}{\partial n}, LL, \frac{\partial \phi_n}{\partial n} \right]^T
\]

The integrals are performed analytically through the azimuthal angle \( \alpha \) from 0 to \( 2\pi \) to yield complete elliptic integrals and then numerically evaluated by using the
Gauss–Legendre quadrature formula (Stroud and Secrest, 1966). The singular integrals are performed by separating the integrals into the logarithmically singular term and the non-singular term (Pozrikidis, 1992). The integral in Eq. (24) becomes

\[
\int_{S_j} \frac{1}{|p_i - q_j|} \, dS = \int_0^1 \frac{d\zeta}{r(\zeta)} \left[ \left( \frac{dr}{d\zeta} \right)^2 + \left( \frac{dz}{d\zeta} \right)^2 \right]^{1/2}
\]

\[
\times \int_0^{2\pi} \frac{d\beta}{\left( (r(\zeta)+r_i)^2 + (z(\zeta)-z_i)^2 \right)^{1/2}}
\]

\[
= \int_0^1 \frac{d\zeta}{r(\zeta)} \left[ \left( \frac{dr}{d\zeta} \right)^2 + \left( \frac{dz}{d\zeta} \right)^2 \right]^{1/2}
\]

\[
\times \int_0^{2\pi} \frac{d\beta}{\left( (r(\zeta)+r_i)^2 + (z(\zeta)-z_i)^2 \right)^{1/2}}
\]

(25)

where \( k^2(\zeta) = \frac{4r_i^2}{(r(\zeta)+r_i)^2 + (z(\zeta)-z_i)^2} \), the \( z \) integral becomes

\[
\int_0^{2\pi} \frac{d\beta}{(1 - k^2 \cos^2 \beta)^{1/2}} = 4 \int_0^{\pi/2} \frac{d\beta}{(1 - k^2 \cos^2 \beta)^{1/2}}
\]

\[
= 4 \int_0^{\pi/2} \frac{d\beta}{(1 - k^2 \sin^2 \beta)^{1/2}}
\]

\[
= 4K(k)
\]

(26)

where \( K(k) \) is the complete elliptic integral of the first kind. Hence we have

\[
\int_{S_j} \frac{1}{|p_i - q_j|} \, dS
\]

\[
= \int_0^1 \frac{d\zeta}{r(\zeta)} \left[ \left( \frac{dr}{d\zeta} \right)^2 + \left( \frac{dz}{d\zeta} \right)^2 \right]^{1/2}
\]

\[
\times \left[ 1 - \frac{4r_i^2}{(r(\zeta)+r_i)^2 + (z(\zeta)-z_i)^2} \right]^{1/2} \cdot K(k)
\]

(27)

Similarly, the integral in Eq. (23) becomes

\[
\int_{S_j} \frac{\partial \phi}{\partial n} \left( \frac{1}{|p_i - q_j|} \right) \, dS
\]

\[
= -4 \int_0^1 \frac{d\zeta}{r(\zeta)} \left[ \left( \frac{dr}{d\zeta} \right)^2 + \left( \frac{dz}{d\zeta} \right)^2 \right]^{1/2}
\]

\[
\times \left[ \left( \frac{dz}{d\zeta} \right) (r(\zeta)+r_i) - \frac{dr}{d\zeta} (z(\zeta)-z_i) - 2r_i \frac{dz}{d\zeta} \right]
\]

\[
\times \left( \frac{E(k)}{1 - k^2(\zeta)} + \frac{2r_i}{k^2(\zeta)} \frac{dz}{d\zeta} \cdot K(k) \right)
\]

(28)

where \( E(k) = \int_0^\pi (1 - k^2 \sin^2 \beta)^{1/2} \, d\beta \) is the complete elliptic integral of the second kind.

Approximations for \( K(k) \) and \( E(k) \) are in the form

\[
K(k) = P(x) - Q(x) \ln(x)
\]

\[
E(k) = R(x) - S(x) \ln(x)
\]

where \( x = 1 - k^2 \).

\( P, Q, R \) and \( S \) are tabulated polynomials that can be found in Abramowitz and Stegun (1965). The matrix equation (22) is solved by standard Gaussian Elimination to yield the normal velocities at the collocation points.

2.6. System of images

A key boundary condition that needs to be specified is the zero-normal-velocity condition on the rigid boundary,

\[
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad z = 0.
\]

(29)

To satisfy this case of a rigid boundary, an image system should be introduced as shown in Fig. 2, whence Green’s function (19) is replaced by

\[
G(p, q) = \frac{1}{|p - q|} + \frac{1}{|p - q'|}
\]

(30)

where \( q' \) is the image of \( q \) in the \( z = 0 \) plane.

2.7. Tangential velocity with cubic spline interpolation

To complete the specification of the surface velocity at a given point, the tangential component must be approximated from the given values of the velocity potential \( \phi(p_i) \) where \( p_i \in S_0 \). We choose the point at the top of the bubble on the
axis of symmetry as the beginning point and arc length as the parametric variable of the velocity potential. Then cubic spline interpolation is used to construct an approximating function $\phi = \phi(s)$ which is used to compute the derivative $\phi$ with respect to $s$.

Two boundary conditions are needed in the calculation. $\frac{\partial \phi}{\partial n} = 0$ at the top of the bubble because of axisymmetry. For the end point $C$, it belongs not only to the bubble surface, but also the rigid boundary of the plate. Since the interface cannot completely detach from the plate and there is no flow through the plate, $C$ can only move along the plate so that the boundary condition (29), with $n$ the normal to the plate, must apply to it. But since it also belongs to the bubble surface, for consistency, the component of the velocity tangential and normal to the bubble surface at $C$ must also add up to a zero velocity in the direction normal to plate, so that (Fig. 3)

$$\left[\frac{\partial \phi}{\partial s}\right]_c \sin \theta - \left[\frac{\partial \phi}{\partial n}\right]_c \cos \theta = 0$$

(31)

where $\theta$ can be identified as the dynamic three-phase contact angle (Liow and Gray, 1988).

### 2.8. Initial condition and non-dimensionalization

We assume that initially the bubble surface is a hemisphere of radius equal to the orifice radius. Gas is pumped into the system at a volumetric flow rate, $Q$, and the initial velocity of the bubble is taken to be

$$U = \frac{Q}{2\pi R_o^2}$$

where $R_o$ is the radius of the orifice.

We use the initial velocity $U$ as a reference velocity and the orifice radius $R_o$ as the reference length. Time is rendered dimensionless with respect to the fundamental timescale $R_o/U$. Thus, the initial condition is that the bubble is a hemisphere with unit dimensionless radius and unit dimensionless velocity.

With respect to velocity potential, the initial conditions are:

$$\phi = -1 \text{ at bubble surface when } t = 0$$

$$\phi = 0 \text{ at } z \to \infty \text{ when } t = 0$$

The initial value of the bubble pressure, $P_b$, is assumed to be the sum of the hydrostatic pressure at the orifice and the pressure due to surface tension,

$$P_b(0) = P_o + \frac{2\sigma}{R_o}. \quad (32)$$

With Eqs. (4), (14) and (32), Eq. (3) can be rewritten as

$$\frac{D\phi}{D\tau} = \frac{1}{2} \left| \vec{u} \right|^2 - gz + \frac{\sigma}{\rho_l} \left( \kappa - \frac{2}{R_o} \right)$$

$$- c_0^2 V_c \rho_l \left[ Q \tau - \hat{V}_b(t) + \hat{V}_b(0) \right]$$

$$+ \frac{\rho_c}{k^2(\pi R_o^2)^3} \rho_l \left[ \left( \frac{d\hat{V}_b}{dt} \right)^2 - \left( \hat{V}_b \right)^2 \right]. \quad (33)$$

In developing the numerical solution of these equations, it is convenient to scale all terms to obtain dimensionless equations. We choose the Froude number, $Fr$, Weber number, $We$, Volume number, $Vn$, and Orifice number, $On$, to represent the physical and geometrical scales in our bubbling system, where:

$$Fr = \frac{U^2}{g R_o}$$

$$We = \frac{\rho_l U^2 R_o}{\sigma}$$

$$Vn = \frac{\rho_c C_a^2 R_o^3}{\rho_l V_c U^2}$$

$$On = \frac{\rho_c}{\rho_l \pi^2 k^2}$$

Thus Eq. (33) becomes

$$\frac{D\hat{\phi}}{D\tau} = \frac{1}{2} \left| \hat{\vec{u}} \right|^2 - \frac{1}{Fr} \hat{\vec{u}}^2 + \frac{1}{We} \left( \hat{\kappa} - 2 \right)$$

$$- Vn \left[ \hat{Q} \tau - \hat{V}_b(t) + \hat{V}_b(0) \right]$$

$$+ On \left[ \left( \frac{d\hat{V}_b}{d\tau} \right)^2 - \left( \hat{V}_b \right)^2 \right] \quad (34)$$

where $\tau$ is the dimensionless unit of time and $\hat{\cdot}$ denotes a dimensionless variable.
2.9. Time stepping

To update the position of the bubble and the potential at its surface through time, an iterative trapezium rule, Euler’s Method, is used

\[
\begin{align*}
\hat{r}_i(\tau + \Delta \tau) &= \hat{r}_i(\tau) + \hat{u}_{i,r}(\tau) \Delta \tau \\
\hat{z}_i(\tau + \Delta \tau) &= \hat{z}_i(\tau) + \hat{u}_{i,z}(\tau) \Delta \tau \\
\hat{\phi}_i(\tau + \Delta \tau) &= \hat{\phi}_i(\tau) + \frac{D\hat{\phi}_i(\tau)}{D\tau} \Delta \tau
\end{align*}
\]

where \(\hat{u}_{i,r}\) and \(\hat{u}_{i,z}\) are radial velocity and axial velocity at \(i\)th point of the bubble surface respectively.

The procedure is summarized in the following steps:

(i) Initialize all variables.
(ii) Increment time by \(\Delta t\).
(iii) The field equation (1) is solved using boundary-integral method. Using the Green’s formula approach the normal velocity is found directly by solving Eq. (22) numerically. The tangential velocity is solved by cubic spline interpolation.
(iv) Use the velocities found in (iii) to update position of the surfaces.
(v) Use the dynamic condition, Eq. (34), to update the surface potentials.
(vi) Go back to (ii) and repeat until detachment happens.

3. Improvements over Hooper’s (1986) model

The proposed model is generally based on Hooper’s (1986) approach to modeling bubble formation using the boundary-integral method. However, we have made several significant improvements and developments in our model.

Firstly, the selection of dimensionless numbers representing the physical and dimensional parameters allows the natural and a priori formulation of the dimensionless equation of motion (Eq. (34)), instead of relying on the questionable iterative method to define length and time scales proposed by Hooper (1986).

Secondly, Hooper (1986) does not explicitly account for the curvature of the bubble surface in relating \(P_l\) to \(P_b\). Instead, it was merely stated that due to numerical instabilities, the surface tension number was always set to zero, thereby ignoring the effect of \(\sigma\) altogether and rendering \(P_l\) equal to \(P_b\) in all the solutions.

Thirdly, the use of a realistic boundary condition (Eq. (31)) at the point where the bubble surface meets the orifice plate allows us to relate the bubble shape to the dynamic three-phase (i.e. gas–liquid–solid) contact angle, \(\theta\), instead of the arbitrary shape criterion introduced in Hooper’s (1986) work.

Furthermore, several material errors in the theoretical development of the earlier model (see Eqs. (31), (A3a), (A4a), (A4b) and (C1) of Hooper, 1986) have been amended in our present work.

4. Result and discussion

The model is verified through comparison between theoretical computation and experimental data for low viscosity liquids as reported in the literature.

For the air-water system of Kupferberg and Jameson (1969) with the following experimental conditions: gas flow rate \(Q = 16.7 \, \text{cm}^3/\text{s}\); the radius of the orifice \(R_o = 0.16 \, \text{cm}\); chamber volume \(V_c = 2250 \, \text{cm}^3\); height of the liquid \(H = 15.24 \, \text{cm}\), the instantaneous shapes and bubble growth curves and chamber pressure fluctuation are shown in Fig. 4. The bubble shapes obtained by the present model are shown in Fig. 4(a), which agree approximately with the experimental shapes shown in Fig. 4(b) obtained from the original photographs in Kupferberg and Jameson (1969). Fig. 4(c) compares the simulated bubble growth curve and chamber pressure fluctuation with the experimental data.

For the CO2–water system of LaNauze and Harris (1974) at elevated pressure with following experimental conditions: gas flow rate \(Q = 10 \, \text{cm}^3/\text{s}\); the radius of the orifice \(R_o = 0.16 \, \text{cm}\); chamber volume \(V_c = 375 \, \text{cm}^3\); height of the liquid \(H = 10 \, \text{cm}\); system pressure \(P_{\infty} = 0.69 \, \text{MN/m}^2\), the instantaneous shapes and bubble volume are shown in Fig. 5. The bubble shapes obtained by the present model are shown in Fig. 5(a). Fig. 5(b) compares the simulated bubble growth curves with the experimental data.

LaNauze and Harris (1974) reported that they observed “double bubbling” under these conditions. The experimental growth curve in Fig. 5(b) shows two distinct periods of growth. This phenomenon can be attributed to the wake effect of preceding bubbles (Zhang and Tan, 2000). Our present model does not account for the wake effect, since the liquid is assumed to be quiescent initially. Nevertheless, the predicted final bubble volume at detachment agrees well with the experimental data, as shown in Fig. 5(b).

Surface tension is one of the contributing factors influencing the bubble volume. To verify the effect of surface tension, data are collected for liquids of different surface tensions, water (\(\sigma = 72.7 \, \text{dyn/cm}\)) and petroleum ether (\(\sigma = 27.1 \, \text{dyn/cm}\)), using the same orifice (\(R_o = 0.2 \, \text{cm}\)) by Davidson and Schüler (1960). Fig. 6 shows the variation of bubble volume at detachment with gas flow rate for different liquids (Davidson and Schüler, 1960, as reported by Ramakrishnan et al., 1969) and the calculated results are compared with the experimental data. It can been seen that at lower gas flow rates, surface tension has little effect on the bubble volume, while at higher flow rates, the lowering of surface tension produces smaller bubbles. It can be noted that our model predictions match the experimental trends closely.

These figures indicate that the results computed by present model are in good agreement with the experimental data.
Fig. 4. (a) Computed sequence of bubble shapes during formation from Kupferberg and Jameson (1969). (b) Approximate experimental shapes from Kupferberg and Jameson (1969). (c) Bubble growth curve and chamber pressure fluctuation from Kupferberg and Jameson (1969).

Fig. 5. (a) Computed sequence of bubble shapes during formation from LaNauze and Harris (1974). (b) Bubble growth curve from LaNauze and Harris (1974).

Fig. 6. Effect of surface tension on bubble size in inviscid liquid from Davidson and Schüler (1960).
5. Conclusion

An improved theoretical model with boundary-integral method has been developed which predicts bubble formation from a single submerged orifice. The model predicts the instantaneous bubble shape, detachment time and liquid circulation around the bubble as well as the chamber and bubble pressure.

Notation

- $c_0$: speed of sound
- $E$: internal energy
- $Fr$: Froude number (dimensionless parameter)
- $g$: acceleration due to gravity
- $H$: height of liquid above the orifice
- $k$: orifice coefficient
- $m$: mass of gas
- $n$: outward normal
- $On$: Orifice number (dimensionless parameter)
- $P$: pressure
- $P_\infty$: system pressure above the bulk liquid
- $q$: gas flow rate through orifice
- $Q$: gas flow rate into chamber
- $r$: radial coordinate
- $R$: gas constant
- $Ro$: orifice radius
- $s$: tangent vector
- $t$: time
- $u$: velocity
- $U$: reference velocity
- $V$: volume
- $V_n$: Volume number (dimensionless parameter)
- $We$: Weber number (dimensionless parameter)
- $z$: axial coordinate

Greek letters

- $\alpha$: azimuthal angle
- $\gamma$: adiabatic exponent
- $\theta$: contact angle
- $\kappa$: curvature
- $\rho$: density
- $\sigma$: surface tension
- $\tau$: dimensionless time
- $\phi$: velocity potential

Subscripts

- $b$: bubble
- $c$: gas chamber

References